NAVAL POSTGRADUATE SCHOOL
Monterey, California

DTIC
S ELECTE D
SEP 0 5 1991
B

A POLYNOMIAL-TIME ALGORITHM FOR COMPUTING THE YOLK IN FIXED DIMENSION
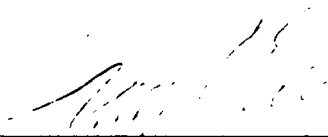
Craig A. Tovey

August 1991

91

NAVAL POSTGRADUATE SCHOOL,
MONTEREY, CALIFORNIA

Rear Admiral R. W. West, Jr.
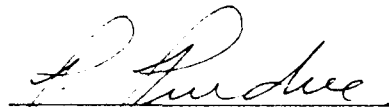Superintendent

Harrison Shull
Provost

This report was prepared by:

_____
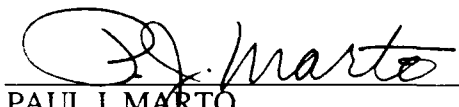CRAIG A. TOVEY
Senior Res. Assoc.—NRC Fellow

Reviewed by:

_____
PETER PURDUE
Professor and Chairman
Department of Operations Research

Released by:

_____
PAUL J. MARTO
Dean of Research

Unclassified
Security Classification of this page

REPORT DOCUMENTATION PAGE

1a Report Security Classification UNCLASSIFIED			1b Restrictive Markings			
2a Security Classification Authority			3 Distribution Availability of Report			
2b Declassification/Downgrading Schedule			Approved for public release; distribution is unlimited			
4 Performing Organization Report Number(s) NPSOR-91-24			5 Monitoring Organization Report Number(s)			
6a Name of Performing Organization Naval Postgraduate School	6b Office Symbol *(If Applicable)* OR		7a Name of Monitoring Organization			
6c Address *(city, state, and ZIP code)* Monterey, CA 93943-5000			7b Address *(city, state, and ZIP code)*			
8a Name of Funding/Sponsoring Organization National Research Council and NSF	8b Office Symbol *(If Applicable)* OR/Tv		9 Procurement Instrument Identification Number			
8c Address *(city, state, and ZIP code)* Washington, DC 20418			10 Source of Funding Numbers			
			Program Element Number	Project No	Task No	Work Unit Accession No

11 Title *(Include Security Classification)* A POLYNOMIAL-TIME ALGORITHM FOR COMPUTING THE YOLK IN FIXED DIMENSION

12 Personal Author(s) Craig A. Tovey

13a Type of Report Technical	13b Time Covered From To	14 Date of Report *(year, month, day)* 1991, August	15 Page Count

16 Supplementary Notation The views expressed in this paper are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

17 Cosati Codes			18 Subject Terms *(continue on reverse if necessary and identify by block number)*
Field	Group	Subgroup	Algorithm; social choice; yolk; voting

19 Abstract *(continue on reverse if necessary and identify by block number)*

The yolk, developed in [16,22], is a key solution concept in the Euclidean spatial model as the region of policies where a dynamic voting game will tend to reside. However, determining the yolk is NP-hard for arbitrary dimension. This paper derives an algorithm to compute the yolk in polynomial time for any fixed dimension.

20 Distribution/Availability of Abstract [X] unclassified/unlimited [ ] same as report [ ] DTIC users	21 Abstract Security Classification Unclassified	
22a Name of Responsible Individual C. A. Tovey	22b Telephone *(Include Area Code)* (408) 646-2140	22c Office Symbol OR/Tv

DD FORM 1473, 84 MAR   83 APR edition may be used until exhausted   security classification of this page

All other editions are obsolete

Unclassified

i

A Polynomial-Time Algorithm for Computing the Yolk in Fixed Dimension

Craig A. Tovey[*]

ISyE and College of Computing[†]

Georgia Institute of Technology

Atlanta Ga 30332

March 12, 1991

revised April 24, 1991; May 12, 1991

Abstract

The yolk, developed in [16,22], is a key solution concept in the Euclidean spatial model as the region of policies where a dynamic voting game will tend to reside. However, determining the yolk is NP-hard for arbitrary dimension. This paper derives an algorithm to compute the yolk in polynomial time for any fixed dimension.

1 introduction

In the Euclidean spatial model [6,3,5,9,34, e.g.], each voter's most preferred policy is represented by an *ideal point* located in $\Re^m$: voters prefer policies (points) closer to their ideal points under the Euclidean norm. This is a widely-studied voting model, with many applications (e.g. [18,31,30,17,36,

1

27,28,29,1]). In this model, a classical equilibrium or core will generally not exist ([32, e.g.]), and in fact the space collapses into chaotic cycles [21,33]. Consequently, much effort has been made in finding a satisfactory alternative to solve the equilibrium problem in the spatial model. The yolk, established in [16,22], has emerged as an important solution concept. Defined as the smallest ball intersecting all median hyperplanes, it is the region of policies where a voting game will tend to stabilize. The yolk is also important by virtue of its close relationships to other solution and evaluation concepts, such as the uncovered set [24,25,12,22], the Pareto set [14], the win set [12], and Shapley-Owen power scores [13].

However, even determining whether the yolk radius is 0 is co-NP-complete in arbitrary dimension [19,2]. Johnson and Preparata [19] give a polynomial algorithm which can be used to determine whether the yolk radius is 0, for any fixed dimension. This suggests the possibility of a polynomial algorithm which computes the yolk, for any fixed dimension. Indeed it had been thought ([22,20,14]) that a linear program in $m+1$ variables and $O(n^m)$ constraints would compute the yolk, but unfortunately it can fail [39].

In this paper we derive a polynomial algorithm for arbitrary fixed dimension. As is typical with pseudopolynomial time algorithms, the time complexity makes the algorithm impractical for moderate or large dimension. At present this does not make for a serious limitation, because nearly all the empirical studies to date have been in two dimensions. Poole and Rosenthal, analyzing 19th and 20th century United States Congressional roll call data, find that 2 dimensions have nearly the same predictive power as higher dimensions [28]. So in at least some applications an efficient algorithm for the two-dimensional case may be all that is ever required.

2 Preliminaries

Let us fix some notation. The voter ideal points are a set $V \subset \Re^m$, where $|V| = n$.

For any hyperplane h in $\Re^m$, we denote the two closed halfspaces defined by h as h^+ and h^-.

A hyperplane h is *median* (with respect to V) iff each closed halfspace

2

it defines contains at least half the voters, that is iff

$$|h^+ \cap V| \geq n/2 \text{ and } |h^- \cap V| \geq n/2.$$

The yolk of V is the smallest ball intersecting all median hyperplanes (see Figure 1).

A *median split* of V is defined to be any pair of sets (S, T) such that:

- $S \cup T = V$;

- $|S| \geq n/2$;

- $|T| \geq n/2$.

A hyperplane $\{x : p \cdot x = p_0\}$ is *consistent* with a median split (S, T) iff $S \subseteq h^+ \cap V$ and $T \subseteq h^- \cap V$. The family of all median splits is denoted $\mathcal{M}$.

The following Proposition is immediate but useful:

Proposition 1: A hyperplane is median iff it is consistent with some median split $(S, T) \in \mathcal{M}$.

Originally [16,22] the yolk was defined only for n odd. Koehler[20] proposed extending it to n even. In this paper n may be odd or even; the algorithm is slightly more complicated in the latter case.

3   what determines the yolk radius?

The main obstacle to an efficient algorithm is that there are infinitely many median hyperplanes. We would like to pass from the infinite to the finite by extracting a small subset of these which determine the yolk. That is, our goal is to identify a crucial "determining" subset of the median hyperplanes, such that the yolk can be determined just from that subset. This goal will be realized by Theorem 1 in this section.

The natural subset to try to use for Theorem 1 would be the set of extremal median hyperplanes. A hyperplane is *extremal* iff it contains m points of V. (In the degenerate case we would require the m points to be affinely independent). In m dimensions, the extremal median hyperplanes are precisely the crucial subset to determine whether or not the yolk radius is 0 [19,35]. For the case $m = 2$ extremal median hyperplanes are known as

3

limiting median lines [22,20], and indeed preliminary algorithmic efforts for the two-dimensional case ([22,20]) were based on the plausible assumption that the limiting median lines determine the yolk.

Unfortunately, this assumption can fail, even in two dimensions ([39]). Thus we have to augment the extremal set to attain a determining set. The set we arrive at in this section is infinite in one sense, but only a little bigger than the extremal set in another sense. We defer the statement of Theorem 1 to the end of the section, because the necessary terminology is developed in the course of its proof.

To develop the determining set, we recapitulate part of the proof of yolk convergence in [40]. There upper bounds are sought on the radius of the yolk, and the main obstacle is again the infinitude of median hyperplanes. The proof strategy is to formulate a mathematical program whose solution is the yolk radius, and then take advantage of certain "nice" properties of the mathematical program.

Let r denote the radius of the yolk of the set V. (Since V is static, we suppress it as an argument.) For any point $x \in \Re^m$, let $r(x)$ denote the radius of the smallest ball centered at x which intersects all the median hyperplanes. We call $r(x)$ the x-centered yolk. Obviously $r = \inf_x r(x)$.

Next we develop a mathematical program for $r(x)$.

The key to the formulation is a kind of duality often termed "polarity" (see e.g. [37,26]) to characterize the set of median hyperplanes. Any hyperplane $\{y : p \cdot y = p_0\}$ is specified by the $(m+1)$-tuple p, p_0. If moreover $\|p\|^2 = 1$ then p_0 is the distance from that hyperplane to the origin. For any particular median split, consistency of p, p_0 with that split is simply expressed as a set of linear inequalities.

In particular, for any median split $(S, T) \in M$ and any point $z \in \Re^m$ the nonlinear program below finds the consistent median hyperplane farthest from z.

$$\max |p \cdot z - p_0| \tag{1}$$

$$\sum_{j=1}^m p_j^2 \quad = 1 \tag{2}$$

$$p \cdot v_i \quad \geq p_0 \quad \forall v_i \in S \tag{3}$$

$$p \cdot v_i \quad \leq p_0 \quad \forall v_i \in T \tag{4}$$

The first constraint (2) normalizes the hyperplane (p, p_0) and the other two

4

linear inequalities (3,4) ensure it is consistent with (S, T).

Now the radius of the x-centered yolk is the distance from x to the farthest median hyperplane. By proposition 1 this is the farthest hyperplane consistent with some median split in $\mathcal{M}$. Therefore, $r(x)$ would occur at the maximum of (1)–(4), taken over $\mathcal{M}$. When we take this maximum, the absolute value signs can be removed from p_0 in the objective function (1), because the maximizing over the median split (T, S) is the same as minimizing over the median split (S, T).

To further simplify notation, we take $x = 0$ in the remainder of the development of Theorem 1. There is no loss of generality since $r(x)$ depends only on the relative locations of the ideal points V to x.

Therefore, the following family of nonlinear mathematical programs determines the value of $r(0)$, the yolk centered at the origin.

$$r(0) = \max_{\mathcal{M}} \quad \max p_0 \; s.t. \tag{5}$$

$$\|p\| = 1 \tag{6}$$

$$p \cdot v_i \geq p_0 \; \forall v_i \in S \tag{7}$$

$$p \cdot v_i \leq p_0 \; \forall v_i \in T \tag{8}$$

This formulation (5–8) does not appear particularly attractive. That the outer maximum is taken over the exponentially large set $\mathcal{M}$ is not promising at first glance.

In [40] the goal is to find upper bounds on $r(0)$. To this end, constraint (6) is relaxed to the set of linear constraints $-1 \leq p_j \leq 1 \forall j = 1, \dots m$. When this is done, a lucky thing happens: the outer maximization is taken over an exponentially large class of linear programs, but all of the linear programs share the identical set of basic solutions, which is only polynomially large. Then the inner and outer maximizations may be replaced by a single maximum, taken over all basic solutions which are feasible in at least one member of the class of linear programs. For (p, p_0) to be feasible in at least one of the linear programs, it must be consistent with at least one of the median splits in $\mathcal{M}$. By Proposition 1, checking this is equivalent to checking if (p, p_0) is median, which is easy to do.

For the purposes of algorithm design, we require an exact formulation rather than an upper bound. The subproblem of the exact formulation is

almost a linear program; just one constraint is quadratic. This suggests that we apply the Karush-Kuhn-Tucker conditions to (5–8), and perhaps a similarly lucky simplification will obtain.

The program for $r(0)$ is:

$$\max_{(S,T) \in \mathcal{M}} \quad \max f(p) = p_0 \ s.t. \tag{9}$$

$$\sum_{i=1}^{m} p_i^2 = 1 \tag{10}$$

$$p \cdot v_i \geq p_0 \ \forall v_i \in S \tag{11}$$

$$p \cdot v_i \leq p_0 \ \forall v_i \in T \tag{12}$$

Apply the KKT conditions to the inner maximization, using multipliers π_0 for the quadratic constraint (10) and π_i for the linear constraints (11,12). When $v_i \in S \cap T$, the two constraints combine to yield an equality. So a single unrestricted (in sign) dual variable π_i can be used for the two constraints in this case. The KKT conditions yield:

$$\sum_{j-1}^{m} p_i^2 = 1; \tag{13}$$

$$p \cdot v_i \geq p_0 \ \forall v_i \in S; \tag{14}$$

$$p \cdot v_i \leq p_0 \ \forall v_i \in T. \tag{15}$$

$$\pi_i(p \cdot v_i - p_0) = 0, \ \forall i \ in S \cup T; \tag{16}$$

$$\nabla f = (0, 0, \ldots, 0, 1) = 2\pi_0(p_1, p_2, \ldots p_m, 0) - \sum_i \pi_i(v_i, -1). \tag{17}$$

The first constraint (13) says p is normalized, $||p||^2 = 1$. The second and third constraints (14,15) say that the hyperplane (p, p_0) is consistent with the median split (S, T). (These three are just primal feasibility). The fourth complementary slackness condition (16) says that the multiplier or dual variable π_i is zero for any point v_i not on the hyperplane (p, p_0). We call the v_i that are on the hyperplane "binding" since their constraints are binding. The last condition (17) is a vector equation and breaks into $\sum_i \pi_i = 1$ and $2\pi_0 p = \sum \pi_i v_i$. Thus this condition says that p is an affine combination of the binding v_i, up to scaling by the $2\pi_0$ factor.

To understand the geometric meaning of the KKT conditions on this problem, first consider the simple case where there is exactly one binding

point, say v_k. Then $p = v_k/2\pi_0$ and $||p|| = 1$ implies $p = v_k/||v_k||$ (and also $2\pi_0 = ||v_k||$). Since v_k is binding, $p \cdot v_k = p_0 = ||v_k||$. So p is simply the vector v_k, scaled to be a unit vector, and p_0 is the distance from the origin to the point v_k. The hyperplane p, p_0 is the hyperplane normal to the vector from 0 to v_k.

More generally (see Figures 2 and 3), suppose there are $l \geq 2$ binding v_i represented by the set B. Then $p = \sum_{i \in B} \pi_i v_i$ where $\sum_{i \in B} \pi_i = 1$. Thus p, scaled by $2\pi_0$, is in the affine hull of the binding points B. The scaling factor is whatever is necessary to keep $||p|| = 1$. Also, $p \cdot v_i = p_0 \ \forall i \in B$. For notational convenience, denote the l points of B as $v_1, \cdots, v_l$. The preceding equation implies $p \cdot v_1 = p \cdot v_2 = \cdots = p \cdot v_l$. So $p \cdot (v_k - v_1) = 0 \forall k = 2, \ldots, l$. Geometrically, p is the vector from 0 to the unique point closest to 0 on the affine hull of the binding v_i.

Taken together, the geometric properties of p imply that p is the normalized normal vector to the $(l-1)$-dimensional affine space of the binding points, in the l-dimensional space defined by the origin and the points of B. That the points of B are binding forces (p, p_0) to be the $(m-1)$-dimensional hyperplane containing this affine hull and normal to p of course. Finally, the second and third constraints require that this hyperplane (p, p_0) be consistent with the split S, T.

The preceding gives the meaning of the KKT conditions for a single split (S, T). Thus the KKT conditions reduce consideration from an infinite set to a finite but exponentially large ($|\mathcal{M}|$) collection of polynomial sized ($|B|$) sets of hyperplanes. When we take the maximum over all median splits, we are lucky as hoped. The only place where (S, T) appears is in the requirement that (p, p_0) be consistent with (S, T). Taken over $\mathcal{M}$, splits, this is transformed into the requirement that (p, p_0) be consistent with at least one median split $(S, T) \in \mathcal{M}$. By Proposition 1, this is equivalent to the requirement that (p, p_0) be a median hyperplane. So the similarity among the subproblems, together with Proposition 1, further reduces consideration to a polynomial number of hyperplanes.

We define the set of *determining hyperplanes* accordingly (see Figures 2 and 3): let $B \subset V$ where $|B| = l \leq m$. Let $Aff(B)$ denote the $(l-1)$-dimensional hyperplane that is the affine hull of B. Let p be the unit vector coincident with the shortest line segment connecting the origin and $Aff(B)$, and let (p, p_0) be the hyperplane containing $Aff(B)$. Then (p, p_0) is a determining hyperplane. We may refer to it as the determining hyper-

7

plane of V specified by B at 0. When V and 0 are clear from context we will simply refer to it as the determining hyperplane specified by B.

If a determining hyperplane is median, it is called a *determining median hyperplane*. We can now state the main result of this section.

Theorem 1. The determining median hyperplanes suffice to determine the radius of the 0-centered yolk. □

We remark that after translation of V Theorem 1 applies to the x-centered yolk for any x. We thus extend the definition of determining hyperplanes to the case of arbitrary x. It is important to see that the determining hyperplanes depend on x. The set of determining hyperplanes is polynomially bounded for any fixed x, although their union is in general infinite.

Notice that the determining hyperplanes specified by sets B where $|B| = m$, are precisely the class of extremal hyperplanes discussed at the beginning of this section. Theorem 1 shows that these are needed to determine the yolk, but they must be augmented by the hyperplanes specified by smaller sets. The extremal hyperplanes are the only ones that are fixed, independent of x. The other ones "swivel" (or pivot) to be normal to x as x changes.

Corollary 1: The radius $r(x)$ of the x-centered yolk can be determined in polynomial time $O(n^{m+1})$, for any fixed dimension m.

Proof: The number of determining hyperplanes at x is the number of choices for B:

$$\sum_{j=1}^{m} \binom{n}{j} = O(n^m).$$

For each choice of B, the hull $Aff(B)$, the determining hyperplane specified by B, and its distance to x obviously can all be computed in $O(1)$ (constant) time (m is fixed). Moreover, whether the determining hyperplane is median can be checked in $O(n)$ time. Thus the distance of the farthest determining median hyperplane to x can be computed in time $O(n^{m+1})$. □

8

4 a polynomial time algorithm for the yolk in fixed dimension

In this section we develop a polynomial algorithm to compute r. We need a couple of lemmata first.

Lemma 1: $r(x)$ is convex and continuous in x.

Proof: Let $\mathcal{H}$ denote the set of all median hyperplanes. (The set $\mathcal{H}$ has infinite cardinality, in general.) For any $h \in \mathcal{H}$ define $d(x, h)$ as the distance from x to h. Then

$$r(x) = \sup_{h \in \mathcal{H}} d(x, h)$$

Now $d(x, h)$ is convex in x for all fixed $h \in \mathcal{H}$. Hence $r(x)$ is convex since the supremum of convex functions is convex. Continuity also follows easily because the functions $d(x, h)$ are a uniformly continuous class.   □

We temporarily make a nondegeneracy assumption, that the points V are in totally general position. Precisely, we assume the following: all subsets $S \subseteq V$, $|S| \le m + 1$, have full affine dimension, *i.e.*, the points in each such S are affinely independent. This enables a simpler development of the algorithm for the nondegenerate case (Lemma 2 and Theorem 1). It will turn out that the same algorithm works correctly in the degenerate case. This extension will be ·reated later (Lemma 3 and Corollary 2).

Lemma 2: Suppose V is nondegenerate and fewer than $m + 1$ determining median hyperplanes are binding at z. If $|V|$ is odd then z is not the yolk center. In the case $|V|$ even, if moreover z is not the midpoint of the shortest line segment connecting two determining median affine hulls $\mathrm{Aff}(B_1)$, $\mathrm{Aff}(B_2)$ where $|B_1| + |B_2| \le m + 1$, then z is not the yolk center.

Proof: Let z satisfy the hypotheses stated for the case $|V|$ odd. We wish to define a direction to move from z along which $r(x)$ is decreasing. Let J denote the set of determining median hyperplanes binding at z. By assumption, $|J| \le m$. Since $|V|$ is odd, there are no parallel median hyperplanes. Thus by nondegeneracy the hyperplanes of J must possess at least one intersection point, say w. Then if we move x from z towards w, we simultaneously move closer towards all the binding median hyperplanes. Intuitively, this should decrease $r(x)$, the distance to the farthest median hyperplane.

9

If all determining hyperplanes were extremal, this intuitior would be rigorously justified with no additional argument. However, from Theorem 1 we know that we must also consider determining hyperplanes specified by sets of points B with $|B| < m$. These hyperplanes are more complicated than the extremal ones, because they vary with x ($Aff(B)$ does not vary with x, but the hyperplane normal to the line segment from x to $Aff(B)$ does). In particular, we must resolve two problems. First, it must be possible to move x a positive distance from z without some set B, whose determining hyperplane is not median at z, suddenly becoming median. For if this happened, the new median hyperplane might be far from x and $r(x)$ might increase. Second, suppose a hyperplane in J were non-extremal ($|B| < m$). In this case, the determining binding median hyperplane specified by B rotates as we move x from z towards w. We must be sure that we actually are moving closer to it, despite its rotation.

To deal with the first of these problems, observe that for any B, the set of points x, at which the determining hyperplane specified by B is median, is closed. Therefore its complement is open. Since z is in the complement, there is an open ball containing z within which the determining hyperplane specified by B is not median. By Theorem 1 the number of such B is finite. The intersection of a finite number of open sets is finite. Therefore, for sufficiently small $\epsilon > 0$, the first problem does not occur at the points $z + \epsilon(w - z)$.

The second problem is easily resolved. Although the determining hyperplane specified by B changes with x, recall that the affine hull $Aff(B)$ does not. Even when x changes, its distance to B's hyperplane is still just the distance to the fixed affine hull $Aff(B)$. Since B's hyperplane is orthogonal to z and w is in the hyperplane, a sufficiently small movement towards w must bring x closer to $Aff(B)$. This completes the case when $|V|$ is odd.

When $|V|$ is even, parallel median hyperplanes may exist. If such a pair is in J then no direction of improvement may exist since it is not possible to move closer to both simultaneously. Therefore we must supplement the proof to deal with the case of parallel determining median hyperplanes. Fortunately, this case can only arise in a very limited number of situations.

Suppose then that J contains two distinct parallel median hyperplanes, generated by B_1 and B_2. First we observe $B_1 \cap B_2 = \phi$ since otherwise their two affine hulls would intersect amd the parallel hyperplanes would not be

distinct. Second, observe that $|B_1| + |B_2| \leq m + 1$, for if $|B_1| + |B_2| \geq m + 2$ then $\dim(\text{Aff}(B_1)) + \dim(\text{Aff}(B_1)) \geq m$ and by the nondegeneracy assumption the two affine hulls would have nonempty intersection.

Third, we claim that there is a unique shortest line segment connecting the affine hulls of B_1 and B_2. For suppose otherwise: let $\overline{xy}$ and $\overline{wz}$ be two shortest segments. We have

$$||x - y|| = ||w - z|| = \min_{u_i \in Aff(B_i)} ||u_1 - u_2|| \neq 0,$$

where $x, w \in Aff(B_1)$ and $y, z \in Aff(B_2)$. The space spanned by x, w, y, and z is at most 3 dimensional and can be visualized easily. Clearly if the two segments $\overline{xy}$ and $\overline{wz}$ are not parallel then $(x + w)/2$ is closer to $\text{Aff}(B_2)$ than x, a contradiction. Thus x, y, w, z are coplanar. Moreover they are the vertices of a rectangle, or y would not be the point in $\text{Aff}(B_2)$ closest to x.

Algebraically, this means $x - w = y - z \neq 0$. Recall that an affine combination is a linear combination in which the sum of the coefficients equals 1. Since x and w are affine combinations of the points of B_1, $x - w$ can be expressed as a nontrivial linear combination

$$x - w = \sum_{i=1}^{|B_1|} \lambda_i r_i; \quad \sum \lambda_i = 0,$$

where $r_i \in B_1$. Similarly $y - z$ can be expressed as a nontrivial linear combination

$$y - z = \sum_{i=|B_1|+1}^{|B_1|+|B_2|} \lambda_i r_i; \quad \sum \lambda_i = 0,$$

where $r_i \in B_2$. The equality of these two expressions implies the nontrivial linear equality

$$\sum_{i=1}^{|B_1|+|B_2|} \lambda_i r_i = 0; \quad \sum \lambda_i = 0,$$

where $r_i \in B_1 \cup B_2$.

Since $|B_1| + |B_2| \leq m + 1$, this equation contradicts the nondegeneracy assumption of affine independence. This verifies the third observation, that there is a unique shortest line segment connecting the two affine hulls.

11

Fourth (and finally), we observe that z must be the midpoint of this shortest line segment. This follows because the line segments from z to each affine hull are of minimum possible and equal length, and are normal to the containing median hyperplanes.

Thus we have characterized the situations in which two determining median hyperplanes from J may be parallel. This completes the proof of Lemma 2. □

The pseudopolynomial time algorithm for computing the yolk is based on Theorem 1 and Lemma 2.

Theorem 2: For any fixed m, the yolk of n nondegenerate points may be computed in polynomial time $O(n^{(m+1)^2})$.

Proof: The number of determining hyperplanes is $O(n^m)$ by Theorem 1. The number of $(m+1)$-tuples of these is $O(n^{m(m+1)})$. For each $(m+1)$-tuple, compute the set of points equidistant to the corresponding set of $Aff(B)$ (this is O(1) since m is fixed). Let the set of all such points be denoted E. Thus $|E| = O(n^{m(m+1)})$.

By Lemma 2, the yolk center is in E. For any point $e \in E$, the radius $r(e)$ may be computed in time $O(n^{m+1})$ by Corollary 1. The point $c \in E$ that minimizes $\{r(e) : e \in E\}$ is the yolk center and $r(c)$ is the yolk radius. The time complexity is $O(|E|n^{m+1}) = O(n^{m(m+1)+m+1}) = O(n^{(m+1)^2})$ as claimed.

In the case $|V|$ even, we must also compute, for every pair $B_1, B_2 :$ $B_1 \cap B_2 = \phi; |B_1| + |B_2| \leq m + 1$, the midpoint of the unique shortest connecting line segment. There are $O(n^{m+1})$ such pairs and the midpoints must be included in the set of potential yolk centers E. The size of E remains $O(n^{m(m+1)})$ and the algorithm complexity is unchanged. □

Now we show that the algorithm of Theorem 2 applies to the degenerate case as well. Lemma 3 shows that slight perturbations of V will result in only slight perturbations to the yolk.

Lemma 3: Let V be any configuration and let $\tilde{V}$ be a nondegenerate infintessimal perturbation of V. Then the radius and center of the yolks of V and $\tilde{V}$ differ only infintessimally. When $|V|$ is even, the center of the yolk of $\tilde{V}$ differs only infintessimally from one of the yolk centers of V.

Proof: Suppose V and $\tilde{V}$ are two configurations that differ infintessimally, so that $|V - \tilde{V}|^\infty < \epsilon$ for suitably small ϵ. Let $r^V(x)$ denote the radius of the

12

x-centered yolk of configuration V. It is obvious that for any x, $r^V(x)$ and $r^{\tilde{V}}(x)$ can only differ infintessimally, since each median hyperplane moves parallel to itself only infintessimally. Thus if we slightly perturb V to get the ideal points in general position, the yolk radius is perturbed only slightly.

However, this does not yet assure that the center of the yolk is only slightly perturbed as well. Let C denote the set of points at which $\min_x r^V(x)$ is attained. That is, C is the set of possible yolk centers.[1] (Note C is convex.) Now for arbitrary $\delta > 0$ let C' denote $\{z : \exists y \in C, \|z - y\| < \delta\}$, the δ-neighborhood of C. Let D denote the closed set which is the complement of C'. By Lemma 1, the minimum of $r^V(x)$ on D is attained at a point q on the boundary of D with C'. Let $\nu = r^V(q) - r^V(c) > 0$. Now select $\epsilon > 0$ so that $\|r^V(x) - r^{\tilde{V}}\|^\infty < \nu/3$. Select any $c \in C$. Then for all $d \in D$, we have $r^{\tilde{V}}(d) \geq r^V(d) - \nu/3 \geq r^V(q) - \nu/3 > r^V(q) - 2\nu/3 = r^V(c) + \nu/3 \geq r^{\tilde{V}}(c)$. Therefore the minimum (actually all the minima) of $r^{\tilde{V}}(x)$ is (are) attained in C', the δ-neighborhood of C. Thus for arbitrary $\delta > 0$ there exist sufficiently small perturbations of V that do not perturb the yolk center by more than δ. This justifies taking V to be a nondegenerate configuration. $\square$

Lemma 3 permits there to be arbitrarily close *nondegenerate* configurations whose centers differ substantially. But for any nondegenerate configuration, there exists a neighborhood of configurations whose centers are close.

Example: Let V be the degenerate configuration of four points as the vertices of a rectangle, $\{(0,0), (2,0), (0.1), (2,1)\}$. Then $C = \{(1, \alpha) : 0 \leq \alpha \leq 1\}$. If V is perturbed infintessimally to nondegenerate $\tilde{V}$, then the center of the yolk of $\tilde{V}$ will be infintessimally close to either $(1,0)$ or $(1,1)$, depending on how the short sides of the perturbed rectangle slant.

Now we extend the algorithm to the degenerate case.

Corollary 2: For any fixed m, the yolk of n points may be computed in polynomial time $O(n^{(m+1)^2})$.

Proof: Let V have yolk center c and radius r. If V were degenerate, it could in principle be infintessimally perturbed to nondegenerate $\tilde{V}$. The algorithm of Theorem 2, applied to $\tilde{V}$, would compute center $\tilde{c}$ and radius $\tilde{r}$. Now if the algorithm of Theorem 2 were applied to V, it would compute

---

[1] If the yolk is unique (it always is when n is odd) then C consists of a single point.

"center" say, c' and "radius" r'. Consider the computations performed by the algorithm: they involve the distances between the points V and hyperplanes detei. ined by these points. Moreover all computations are confined to the compact region defined by the convex hull of V. Therefore, c' and r' must be arbitrarily close to $\bar{c}$ and $\bar{r}$, respectively, for small enough perturbation. But by Lemma 3, the latter are arbitrarily close to c and r, the true yolk center and radius, respectively. Thus cl and r' must in fact be the true values. $\square$

The algorithm of Theorem 2 is very easy to parallelize, because it simply takes the minimum of a large number of independent calculations. We formalize this observation in a corollary.

Corollary 3: For any fixed dimension, the yolk of V may be computed in logarithmic time with a polynomial number of processors.

Proof: The set of equidistant points E may be computed independently in parallel for each $(m + 1)$-tuple described in Theorem 2. The minimum of a polynomially bounded set of numbers may be computed in logarithmic time. Also, the splitting up of the tuples requires only logarithmic time.

It remains to show that for each equidistant point e, the radius of the e-centered yolk, $r(e)$, may be computed in logarithmic time. In the proof of Corollary 1, each choice of B could be processed independently in parallel (assuming sufficiently many processors). The first part of the processing is the computation of $Aff(B)$ and its distance to e: this requires $O(1)$ time (in fixed dimension). The second part of the processing is the check to see if the determining hyperplane is median or not. This may be accomplished in logarithmic time with n processors: each processor handles one point and checks in constant time which halfspace (possibly both) the point is in; the sums are tallied up in logarithmic time.

The algorithm therefore requires only logarithmic time and $O(n^{(m+1)^2})$ processors. $\square$

In practice, with current-day hardware, far less than this full amount of parallelism would be implementable. It is clear from the proof of Corollary 3 that a nearly perfect speed-up could be obtained with any small number of processors.

Even for $m = 2$ dimensions, the complexity estimate of Theorem 2 is too large to ensure practicality. In the next section we make some improvements

14

in the algorithm and in the complexity estimation, to achieve a practical method for problems of moderate size in two dimensions.

5   improvements in performance and time bounds

In this section we improve the actual efficiency and sharpen the analysis of the algorithm for the two-dimensional case, to achieve $O(n^{4.5})$ provable worst-case time complexity. If a conjecture of Erdős, Lovasz, Simmons, and Straus holds, the worst-case complexity reduces to $o(n^{3+\epsilon})$ for all $\epsilon > 0$. Moreover, based on empirical observation, we would expect the algorithm to require $O(n^3 \log n)$ time. We also suggest a variant which could possibly perform in sub-cubic time (for 2 dimensions), though rigorous bounds of this quality remain an open problem.

The first improvement has to do with computing the set of points equidistant to the affine hulls $Aff(B)$. When all three hulls are lines, or all points, there is only one equidistant point, defined by the intersection of two lines. But if some of the hulls are points and some are lines, there will be two equidistant points, defined by the intersection of a line and a parabola. In higher dimensions there may be many ($O(1)$ for any fixed m) equidistant points. In the previous section, we computed $r(x)$ for each of these points. But this is not necessary. By the convexity of $d(x, h)$ in Lemma 1, the distances to the hulls on any convex combination of the equidistant points are less than or equal to the convex combination of the distances at the equidistant points. Therefore, if these hulls are the binding (farthest) from an equidistant point p, and another equidistant point q is closer to the hulls, then p cannot be the yolk center. This means that whenever there are multiple equidistant points, we may discard all but the closest of these.

Now consider a point q, which we suppose is the closest equidistant point to three ($m + 1$ in general) affine hulls, at distance k. Suppose also that the determining lines (hyperplanes in general) are median. Then the yolk radius must be at least k, because the triangle (simplex) formed by 3 ($m + 1$) determining median lines (hyperplanes) has an inscribed circle (ball) of radius k.

Thus each 3-tuple (m-tuple) of affine hulls provides a lower bound on the yolk radius, if the determining lines are median. In the previous section,

we computed $r(q)$ for each equidistant point q, and selected the *smallest* to get the yolk radius. Instead, we discard all but the closest in each set of equidistant points; we further discard any of these whose containing lines are not all medians; then we select the point among those remaining at *largest* distance k from the affine hulls it is equidistant to.

The revised algorithm operates as follows: there are $O(n^{m(m+1)})$ $(m+1)$-tuples of affine hulls. For each of these, use $O(1)$ time to find the closest equidistant point and compute the determining hyperplanes, and $O(n)$ time to check if these hyperplanes are all median. Selecting the largest resulting distance takes no extra time. The resulting complexity is $O(n^{m^2+m+1})$.

The computation taking the most time in the revised algorithm is the check to see if the determining hyperplanes are all median. If an affine hull is extremal then it is its own determining hyperplane (it doesn't "swivel"). So the determining hyperplane is either median or not, independent of which $(m + 1)$-tuple is being considered. It is more efficient to precompute, once for each extremal hyperplane, whether it is median or not. There are only $O(n^{m(m+1)-1})$ $(m + 1)$-tuples of affine hulls in which not all the affine hulls are m-dimensional (i.e. extremal, limiting). For these tuples the processing time is still $O(n)$; for the $O(n^{m(m+1)})$-tuples of extremal hyperplanes the precomputations reduce the processing time to $O(1)$. The resulting complexity is $O(n^{m^2+m})$.

In the two-dimensional case an alternative to preprocessing the extremal hyperplanes is to preprocess the ideal points V so as to be able to query whether a line is median in time better than $O(n)$. If $O(n^2 \log n)$ time and $O(n^2)$ space are used, each such query can be answered in $O(\log n)$ time [7]. This would result in a complexity of $O(n^{m^2+m} \log n)$.

If the two kinds of preprocessing are both used, the resulting complexity is still $O(n^{m^2+m})$, the dominant term due to processing $O(n^{m^2+m})$ tuples of extremal hyperplanes each in constant time. So the best 2-D algorithm we have so far has complexity $O(n^6)$.

To further reduce the time complexity, notice that not all of the extremal hyperplanes will be median. Computational experience makes one suspect that there cannot be a great many median hyperplanes. In two dimensions I have never encountered a configuration from actual data with more than than $2n$ distinct extremal median hyperplanes[2]. Let $e(V)$ denote

---

[2] This is consistent with the experience reported to me by other researchers.

the number of distinct extremal median hyperplanes of the configuration V, and let ϵ_n denote the maximum of $e(V)$ over all nondegenerate configurations $V : |V| = n$. The best known upper bound on e_n in two dimensions is given by Erdős, Lovasz, Simmons, and Strauss:

Theorem 3 [10]: $e_n = O(n^{1.5})$. $\square$

Erdős *et al.* [10], (and independently Edelsbrunner and Welzl [8] [3]) construct configurations V with $e(V) \sim n \log n$. They add, "it appears likely that the lower bound obtained for e_n is closer to the truth than the upper bound ... we conjecture than $e_n = o(n^{1+\epsilon})$ for all $\epsilon > 0$" (p. 149).

We combine Theorem 3 with Theorem 2 and the algorithm modifications discussed in this section, to state improved bounds for yolk computations in two dimensions:

Corollary 4: In two dimensions the yolk may be computed in time $O(n^{4.5})$. If the Erdős-Lovasz-Simmons-Strauss conjecture holds, then the computation may be achieved in time $o(n^{3+\epsilon})$ for all $\epsilon > 0$.

Proof: By Theorem 3 there are $O(n^{4.5})$ 3-tuples of extremal median lines. With the preprocessing of extremal median lines as discussed, these may be processed in time $O(1)$ each. There are $O(n^{1.5+1.5+1}) = O(n^4)$ 3-tuples, each comprised of two extremal median lines and one point ($|B| = 1$). With the preprocessing of V for halfspace queries from [7], the median queries will require $O(\log n)$ time each. So these 3-tuples contribute $O(n^4 \log n)$ to the total processing time. The other kinds of 3-tuples will require only $O(n^{3.5} \log n)$ and $O(n^3 \log n)$ time respectively. Thus the total processing time required is $O(n^{4.5} + n^4 \log n) = O(n^{4.5})$ as claimed.

Now suppose the ELSS conjecture is true. If $e_n = o(n^{1+\delta})$ then there are $o(n^{3+3\delta})$ 3-tuples of extremal median lines to be processed in $O(1)$ time each. There are $o(n^{3+2\delta})$ 3-tuples (of two lines and one point each) to be processed in $O(\log n) = o(n^\delta)$ time each. The other kinds of 3-tuples contribute less. The overall complexity is $o(n^{3+\epsilon})$ where $\epsilon = 3\delta$. $\square$

We remark that if in practice $e(V) \sim n$, as so far observed, then the observed time complexity will be $O(n^3 \log n)$. In this situation the dominant term would be from the $\sim n^3$ 3-tuples consisting of 2 extremal lines and one

---

[3]the upper bound of $O(n^{1.5})$ in [8] on distinct feasible median splits does not give an upper bound on e_n because many lines may be consistent with the same split.

point each. The processing time would be $O(\log n)$ per 3-tuple, to check if the determining line of the point is median.

For general dimension, finding good upper bounds on $e(V)$ remains an open extremal combinatorial problem. Erdős *et al.* remark (p. 149) that their upper bound does not generalize easily. Here we prove an $O(n^{m-.5})$ upper bound on the *expected* value of $e(V)$, in m dimensions.

Proposition 2: Let the n points of configuration V be sampled independently from any nondegenerate distribution μ on $\Re^m$. Then the expected number of extremal median hyperplanes, $e(V)$, is $O(n^{m-.5})$.

Proof: Let $v_1, \ldots, v_n$ denote the points. Let h be the extremal hyperplane specified by $v_1, \ldots, v_m$. Suppose $v_1, \ldots, v_m$ were sampled first. Then h would be fixed, and we could compute $\mu(h^+)$, the probability that a point sampled randomly from μ falls in one of the halfspaces defined by h. Since μ is nondegenerate, $\mu(h) = 0$. Now h will turn out to be median iff exactly half the remaining $n - m$ points fall in h^+. The probability of this event is the probability that, of $n - m$ identical independent Bernoulli trials, each with parameter $p = \mu(h^+)$, exactly half are successful. This probability is bounded by the case $p = 1/2$. In this case the probability, from the binomial distribution, is $2^{m-n} \binom{n-m}{\frac{n-m}{2}} \sim 1/\sqrt{n}$. Therefore, the probability h is median is at most $\sim 1/\sqrt{n}$.

By the linearity of expectation [11], the expected number of median extremal hyperplanes is $O(n^{m-.5})$. $\square$

In the proof of Proposition 2, if $\mu(h) \neq p$, then the probability h is median is smaller than the estimate. The slack in the proof leads me to conjecture that the bound of Proposition 2 holds in the worst case.

Applying Proposition 2 to the general case gives $O(n^{(m+1)(m-.5)})$ $(m+1)$-tuples of extremal median hyperplanes. These can be processed in $O(1)$ time each as before. Also there are $O(n^{(m(m-.5)+m-1)})$ $(m + 1)$-tuples of m extremal median hyperplanes plus one determining hyperplane specified by $B : |B| = m - 1$. These can be processed in $O(\log n)$ time each with the preprocessing of V. The other families of tuples are lower order and do not affect the analysis. The overall (expected) complexity reduces to $O(n^{m^2+m/2-.5})$.

To summarize, the algorithm efficiency can be improved with appropriate preprocessing. The efficiency also depends on the magnitude of

$e(V)$. For two dimensions, the proven time complexity is $O(n^{4.5})$, with a conjecture-based bound slightly worse than cubic.

Finally it should be observed that none of the versions of the algorithm presented so far has very strong use of the convexity of $r(x)$ from Lemma 1. If extremal hyperplanes sufficed to determine the yolk, then as McKelvey has observed [22] the yolk of V could be calculated by solving an $(m + 1)$-variable linear program in $e(V)$ constraints. The algorithm here performs the analog of enumerating all or most of the basic feasible solutions. An analogy with the operation of the simplex method on a min-max linear program suggests the following algorithm: (i) preprocess extremal hyperplanes and V; (ii) solve the linear program of [22] to get a "hot start" initial solution x equidistant by amount d to the affine hulls of sets $B_1, \ldots, B_{m+1}$; (iii) compute for the initial solution the sum of the violations, i.e. the sum over all determining median hyperplanes h at x, of $[d(x,h)]^+$; (iv) while the sum of the violations exceeds 0, iteratively improve the current solution by exchanging one of the B_i for another B, where improvement is measured by the sum of the violations. With $O(n^m)$ choices for B, and the computation of the sum of violations requiring $O(n^m)$, such an algorithm might perform in about $O(n^{2m})$, as compared with about $O(n^{m^2})$ for the enumerative algorithm. In the two dimensional case the algorithm would require $O(n^3)$ ($O(n^{2+\epsilon})$ conjectured) time per iteration. A theoretical analysis of this algorithm, coupled with a proof of the ELSS conjecture, might attain a sub-cubic time complexity. (Notice a sublinear number of iterations would be required.) While this might not improve much on $n^{3+\epsilon}$ for the two dimensional case, it could make yolk computations in three dimensions a practical possibility.

6 Acknowledgments

References

[1] Alesina, Alberto, and Rosenthal, Howard (1991). "Mediating elections"

[2] Bartholdi, J., Narasimhan, N., Tovey, C. (1989). Recognizing majority-rule equilibrium in spatial voting games. To appear in *Social Choice and Welfare*.

[3] Black, D. (1958). *The Theory of Committees and Elections*, Cambridge University Press, Cambridge.

[4] Davis, Otto A., Morris H. DeGroot, and Melvin J. Hinich. "Social Preference Orderings and Majority Rule." *Econometrica* **40** (1972) pp. 147–157.

[5] Davis, Otto A., Hinich, Melvin J., "A mathematical model of policy formation in a democratic society", in Berndt, J.L. (ed), *Mathematical Applications in Political Science II*, Southern Methodist University, Dallas.

[6] Downs, A. (1957). *Economic Theory of Democracy*. Harper and Row, New York.

[7] Edelsbrunner, H., D.G. Kirkpatrick, H.A.Maurer (1982). "Polygonal intersection searching", *Information Processing Letters* **14** pp. 74–79.

[8] Edelsbrunner, H., E. Welzl (1985). "On the number of line-separations of a finite set in the plane" *Journal of Combinatorial Theory Ser. A* **38** pp. 15–29.

[9] Enelow, James, and Hinich, Melvin (1984). *The Spatial Theory of Voting*, Cambridge University Press, Cambridge.

[10] Erdös, P., L. Lovasz, A. Simmons, E.G. Straus (1973). "Dissection graphs of planar point sets", in *A Survey of Combinatorial Theory*, J. N. Srivastava *et al.* eds., North-Holland, Amsterdam, pp. 139–149.

[11] Erdös, P., and Spencer, J (1974). *Probabilistic methods in Combinatorics*, Academic Press, New York.

[12] Feld, S.L., B. Grofman, Richard Hartley, M. O. Kilgour, N. R. Miller (1987). "The uncovered set in spatial voting games", *Theory and Decision* **23** pp. 129–156.

[13] Feld, SL, Grofman, B. (1990). "A theorem connecting Shapley-Owen power scores and the radius of the yolk in two dimensions," *Social Choice and Welfare* **7** 71–74.

[14] Feld, S.L., B. Grofman, N.R.Miller, "Centripetal forces in spatial voting: on the size of the yolk," *Public Choice* **59** (1988) 37–50.

[15] Feld, S.L., B. Grofman, N.R.Miller (1989). "Limits on agenda control in spatial voting games," *Math Model* **12** pp. 405–416.

[16] Ferejohn, John A., Richard McKelvey, and Edward Packel (1984). "Limiting Distributions for Continuous State Markov Models", *Social Choice and Welfare* **1** pp. 45–67.

[17] Grofman B, Norrander B, Feld SL (1988). U.S. electoral coalitions, 1956–1984. Presented at the Annual Meeting of the American Political Science Association, Washington, D.C. August 30 – September 3.

[18] Hinich, Melvin (1978). "Some evidence on non-voting models in the spatial theory of electoral competition," *Public Choice* **33** pp. 83–102.

[19] Johnson, David S., Preparata, F.P. (1978). "The densest hemisphere problem ", *Theor. Comput. Sci.* **6**, pp. 93–107.

[20] Koehler, D.H. The Size of the Yolk: Computations for Odd and Even-Numbered Committees. *Social Choice and Welfare* **7** (1990) 231–245.

[21] McKelvey, Richard D (1979). "General conditions for global intransitivies in formal voting models," *Econometrica* **47** 472–482.

[22] McKelvey, Richard D. "Covering, dominance, and institution free properties of social choice," *American Journal Political Science* **30** (1986) 283–314.

[23] McKelvey, Richard D., and Richard E. Wendell. "Voting equilibria in multidimensional choice spaces," *Mathematics of Operations Research* **1** (1976), pages 144-158.

[24] Miller NR. A new solution set for tournaments and majority voting. *Am J Polit Sci* **24** (1980) 68–96.

[25] Miller NR (1983). The covering relation in tournaments: two corrections. *Am J Political Science* **27** 382–385.

[26] Nemhauser, George, and Wolsey, Lawrence *Integer Programming* 1988.

[27] Poole, Keith, and Rosenthal, Howard (1985). "A spatial model for legislative roii call analysis," *American Journal of Political Science* **29** pp. 357–384.

[28] Poole, Keith, and Rosenthal, Howard (1990). "Patterns of Congressional Voting", to appear in *American Journal Political Science*

[29] Poole, Keith, and Rosenthal, Howard (1990). "Spatial Realiignment and the Mapping of Issues in American History: the Evidence from Roll Call Voting," GSIA WP 1990-41, Carnegie-Mellon University, Pittsburgh, PA 15213.

[30] Rabinowitz G, McDonald E. (1986). The power of the siates in U.S. presidential elections. *American Political Science Revew* **80** pp. 65–89.

[31] Rapoport A, Golan E (1985) "Assessment of political power in the Israeli Knessett," *American Political Science Review* **79** pp. 673–692.

[32] Rubenstein, Ariel (1981). "A note about the nowhere denseness of societies having an equilibrium under majority rule," *Econometrica* **47** pp. 511–514.

[33] Schofield, Norman (1978). Instability of Simple Dynamic Games. *Review of Economic Studies* **45** 575–594.

[34] Schofield, Norman (1985). *Social Choice and Democracy.* Springer-Verlag, Berlin.

[35] Schofiela, Norman, Tovey, C. (1991) "On the 50 + ϵ % rule", manuscript, 1991.

[36] Shapley LS, Owen G (1989). Optimal location of candidates in ideological space. *International Journal of Game Theory* **1** 125–142.

[37] Schrijver, A (1987). *Integer Programming.*

[38] Shepsie KA, Weingast BR (1984). Uncovered sets and sophisticated voting outcomes with implications for agenda institutions. *Am J Polit Sci* **28** pp. 49–74.

[39] Stone, Richard E., Tovey Craig A.,"Limiting median lines do not suffice to determine the yolk," December 1990 (to appear in *Social Choice and Welfare*).

[40] Tovey, Craig A., "The almost surely shrinking yolk," October 1990, submitted to *Mathematics of Operations Research.*
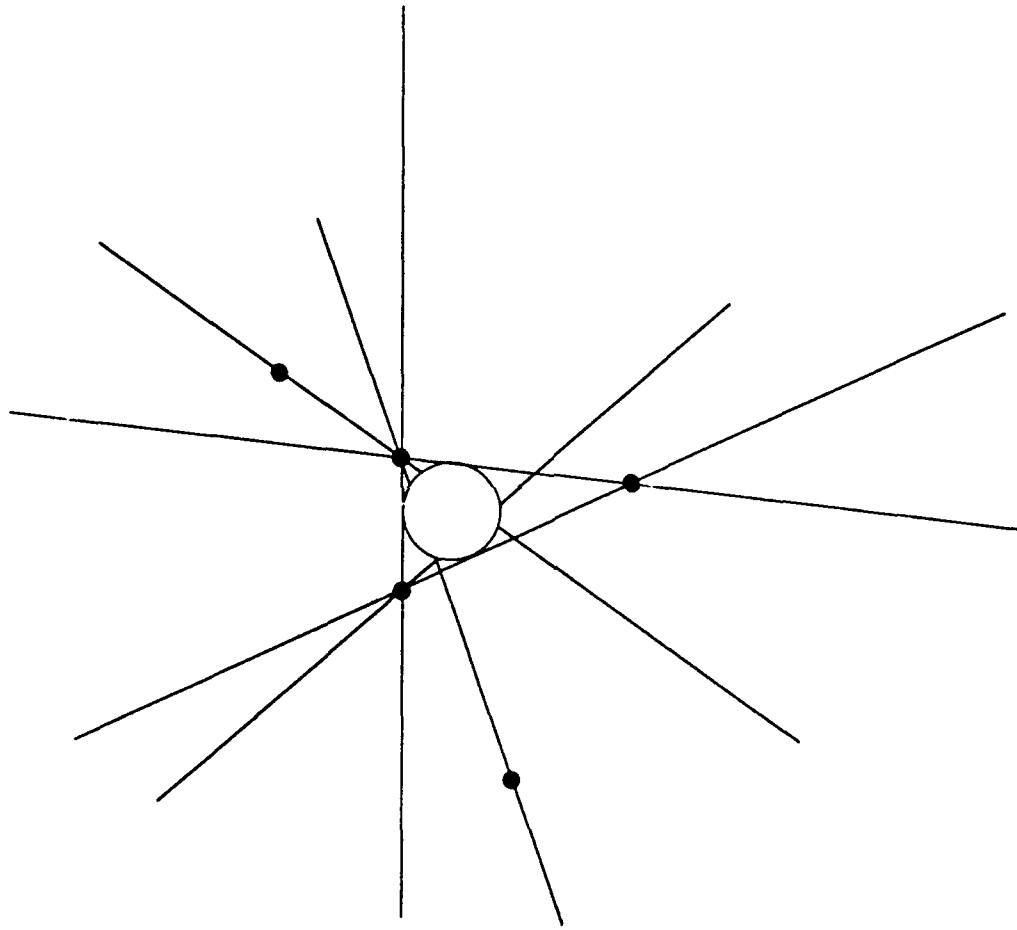
Figure 1. The Yolk
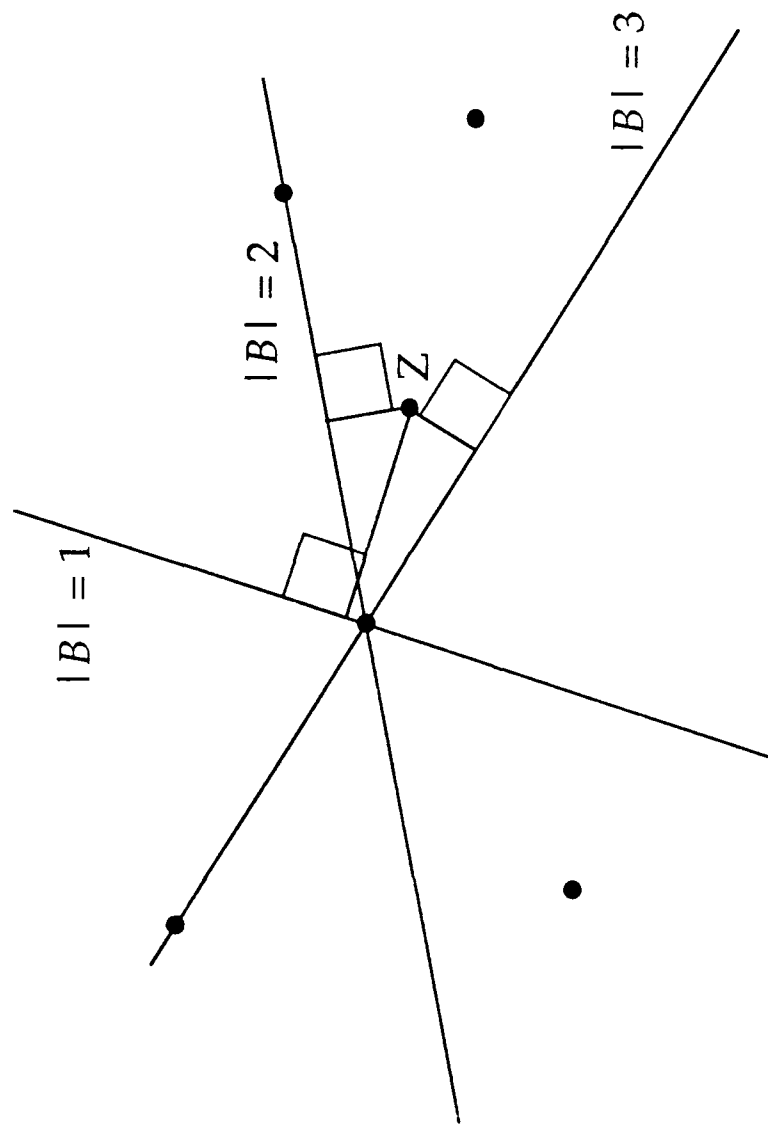
Figure 2. Determining Median Lines in 2 Dimensions
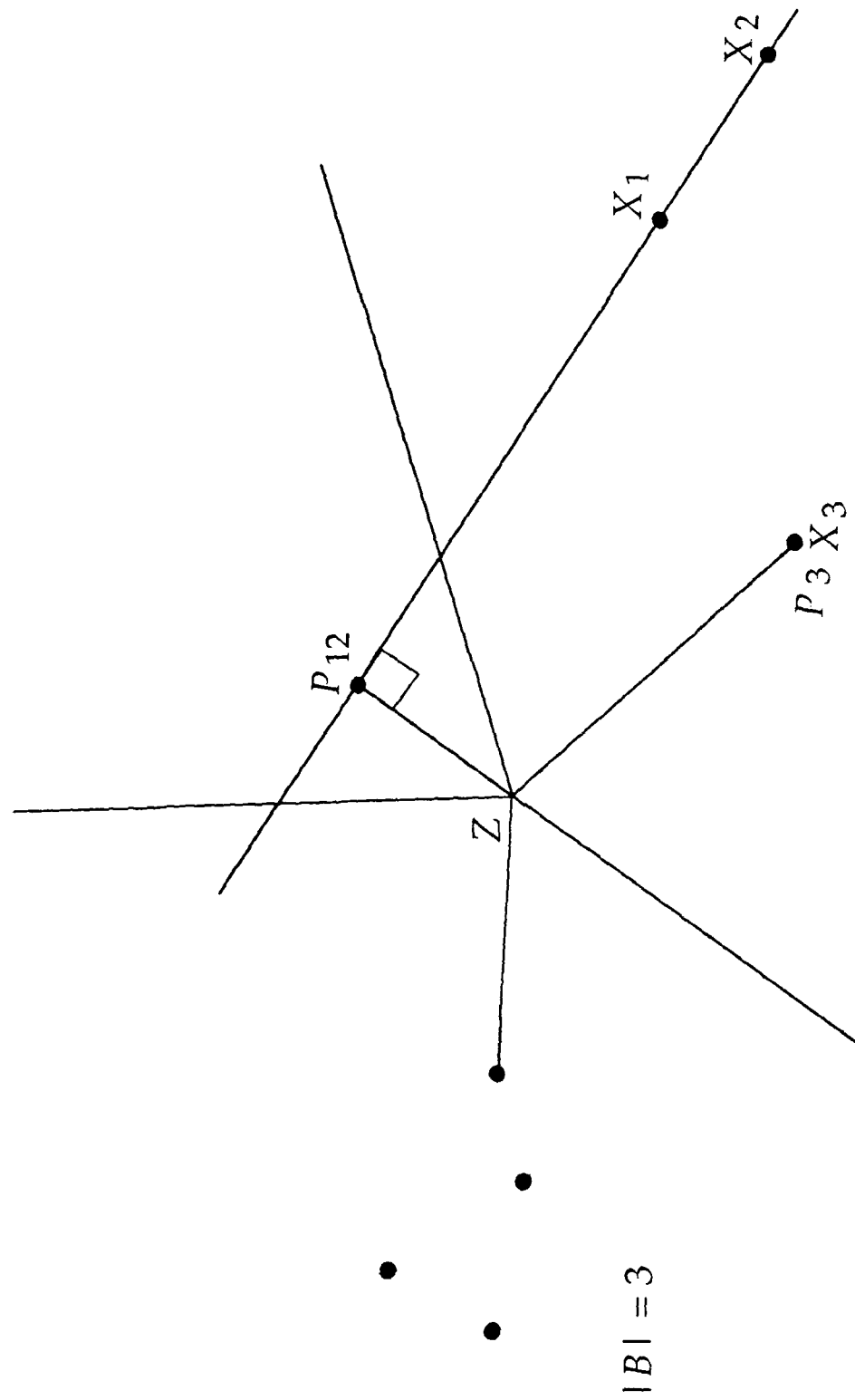
Figure 3. Determining Median Hyperplanes
Three Dimensions

INITIAL DISTRIBUTION LIST

1. Library (Code 0142)..2
   Naval Postgraduate School
   Monterey, CA 93943-5000

2. Defense Technical Information Center...2
   Cameron Station
   Alexandria, VA 22314

3. Office of Research Administration (Code 012) 1
   Naval Postgraduate School
   Monterey, CA 93943-5000

4. Prof. Peter Purdue..1
   Code OR-Pd
   Naval Postgraduate School
   Monterey, CA 93943-5000

5. Department of Operations Research (Code 55).........................1
   Naval Postgraduate School
   Monterey, CA 93943-5000

6. Prof. Craig A. Tovey, Code OR-Tv 20
   Naval Postgraduate School
   Monterey, CA 93943-5000

7. Prof. Lyn R. Whitaker, Code OR-Wh.......................................1
   Naval Postgraduate School
   Monterey, CA 93943-5000

8. Prof. Donald P. Gaver, Code OR-Gv 1
   Naval Postgraduate School
   Monterey, CA 93943-5000

9. Prof. Patricia A. Jacobs, Code OR-Jc 1
   Naval Postgraduate School
   Monterey, CA 93943-5000

10. Prof. Siriphong Lawphongpanich, Code OR-Lp........................1
    Naval Postgraduate School
    Monterey, CA 93943-5000

42. Gideon Weiss...1
ISYE
Georgia Institute of Technology
Atlanta, GA 30332

43. Robert Foley 1
ISYE
Georgia Institute of Technology
Atlanta, GA 30332

44. Loren Platzman...1
ISYE
Georgia Institute of Technology
Atlanta, GA 30332

45. Kenneth Arrow...1
Department of Economics,
Stanford University,
Stanford, CA 94305

46. John Gilmour..1
Center for Political Economoy
Washington University
St. Louis, MO 63130

47. Norman Schofield...1
Center for Political Economoy
Washington University
St. Louis, MO 63130

48. Richard McKelvey..1
Division of Humanities and Social Sciences,
California Institue of Technology,
Pasadena, CA 91125

49. Andrew Caplin..1
Department of Economics
Columbia University,
116th and Amsterdam Ave.
New York, NY 10027